

# A Green and Naghdi Model in a Two-Dimensional Thermoelastic Diffusion Problem for a Half Space

Ibrahim A. Abbas<sup>1, 2, \*</sup>, Marin Marin<sup>3</sup>, Elbaz I. Abouelmagd<sup>1</sup>, and Rajneesh Kumar<sup>4</sup>

<sup>1</sup> Department of Mathematics, Faculty of Science and Arts-Khulais, King Abdulaziz University, Jeddah, Saudi Arabia <sup>2</sup> Department of Mathematics, Faculty of Science, Sohag University, Sohag, Egypt

<sup>3</sup>Department of Mathematics, reading of Science, Transilvania, University of Brasov, Romania

<sup>4</sup>Department of Mathematics, Kurukshetra University, Kurukshetra, India

The present investigation is concerned with the wave propagation of generalized thermoelastic diffusion on a half space with energy dissipation. The surface of the half-space is taken to be traction free and a thermal shock applied on the surface. The problem has been solved numerically using a finite element method. The expressions for displacement components, stresses, temperature, concentration and chemical potential so obtained in the physical domain are computed numerically and illustrated graphically at different times. Comparisons are made with the presence and absence of diffusion.

Keywords: Generalized Thermoelastic Diffusion, Green and Naghdi Model, Finite Element.

## 1. INTRODUCTION

The study of diffusion phenomenon is of great deal of interest due to its many applications in geophysics and industrial applications. Diffusion can be defined as the movement of particles from an area of high concentration to an area of lower concentration until equilibriums reached. Study of the phenomenon of diffusion is used to improve the conditions of oil extractions (seeking ways of more efficiently recovering oil from oil deposits). These days, oil companies are interested in the process of thermoelastic diffusion for more efficient extraction of oil from oil deposits. The governing equations for the displacement and temperature fields as given by the linear dynamical theory of thermoelasticity, introduced by Biot<sup>1</sup> consist of the two coupled partial differential equations. The displacement field is governed by a wavetype equation and the temperature field is governed by a diffusion type equation. The properties of the later are such that a portion of the solution extends to infinity. The theory of couple thermoelasticity was extended by Lord and Shulman (LS)<sup>2</sup> and Green and Lindsay<sup>3</sup> by including the thermal relaxation time in constitutive relations. The theory was extended for anisotropic body by Dhaliwal and Sherief.<sup>4</sup> Green and Naghdi<sup>5</sup> proposed a new generalized thermoelasticity theory by including the thermal-displacement gradient among the independent

are contained in Refs. [6, 7]. Nowacki<sup>8-11</sup> developed the theory of thermoelastic diffusion. In this theory, the coupled thermoelastic model is used. Sherief et al.<sup>12</sup> developed the generalized theory of thermoelastic diffusion with one relaxation time, which allows the finite speeds of propagation of waves. Sherief and Saleh<sup>13</sup> investigated the problem of a thermoelastic half-space in the context of the theory of generalized thermoelastic diffusion with one relaxation time. Singh<sup>14, 15</sup> discussed the reflection phenomena of waves from free surface of an elastic solid with generalized thermodiffusion. Sharma et al.<sup>16</sup> discussed the plane strain deformation in generalized thermoelastic diffusion. Recently, Sherief, and El-Maghraby<sup>17</sup> studied the thermoelastic-diffusion interactions in a thick plate problem. The exact solution of the governing equations of the generalized thermoelastic diffusion theory exists only for very special and simple initial and boundary problems. To calculate the solution of general problems, a numerical solution technique is used. For this reason the finite element method is chosen. Abbas and Abbas et al.18-24 applied the finite element method in different thermoelasticity problems. Recently,25-27 variants problems in waves are studied. Other forms are described for example in the Refs. [28–30].

constitutive variables. An important feature of this theory, which is not present in other thermoelasticity theo-

ries, is that it does not accommodate dissipation of thermal

energy. The relevant fundamental aspects of this theory

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<sup>\*</sup>Author to whom correspondence should be addressed.

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In the present contribution, the two-dimensional problem of generalized thermoelastic diffusion for a half-space is studied. The problem has been solved using generalized thermoelasticity theory proposed by Green and Naghdi theory. The governing equations are solved by finite element method. Numerical results for the displacement components, stresses, temperature, concentration and chemical potential are given and illustrated graphically.

#### 2. BASIC EQUATIONS

According to Green and Nagdhi,<sup>5</sup> Sherief et al.,<sup>12</sup> the governing equations for isotropic, homogeneous solid with generalized thermoelastic diffusion with energy dissipation in the absence of body forces and heat sources are:

The equations of motion

$$\mu u_{i,ij} + (\lambda + \mu)u_{j,ij} - \gamma \theta_i - \beta C_i = \rho \frac{\partial^2 u_i}{\partial t^2} \qquad (1)$$

The equation of heat conduction

$$k^*\theta_{ii} + k\frac{\partial\theta_{ii}}{\partial t} = \rho c_e \frac{\partial^2 \theta}{\partial t^2} + aT_0 \frac{\partial^2 C}{\partial t^2} + T_0 \gamma \frac{\partial^2 e}{\partial t^2}$$
(2)

The equation of mass diffusion

$$DbC_{ii} - Da\theta_{ii} - D\beta e_{ii} = \frac{\partial C}{\partial t}$$
(3)

The constitutive equations

$$\sigma_{ij} = 2\mu e_{ij} + \delta_{ij} (\lambda e_{kk} - \gamma \theta - \beta C)$$

$$P = -\beta e_{kk} + bC - a\theta$$

$$e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})$$
(4)

where list of symbols are given in the nomenclature.

### 3. FORMULATION OF THE PROBLEM

We consider a homogeneous, isotropic, generalized thermodiffusive elastic half space initially at uniform temperature  $T_0$ . We use a fixed Cartesian coordinate system (x, y, z) with origin on the surface x = 0, which is stress free and with y-axis directed vertically into the medium. The region x > 0 is occupied by the elastic solid with generalized thermodiffusion. We restrict our analysis parallel to xy-plane. We assume that all quantities are functions of the coordinates x, y and time t and independent of coordinate z. So the components of displacement vector, temperature and concentration can be taken in the following form

$$u = u_x = u(x, y, t), \quad v = u_y = v(x, y, t)$$
  

$$w = u_z = 0, \quad T = T(x, y, t), \quad C = C(x, y, t)$$
(5)

From Eqs. (4) and (5), we can obtain the constitutive equations

$$e_{xx} = \frac{\partial u}{\partial x}, \quad e_{yy} = \frac{\partial v}{\partial x}, \quad e_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$
  
 $e_{zz} = 0, \quad e_{xz} = 0, \quad e_{yz} = 0$  (6)

 $\sigma_{xx} = 2\mu \frac{\partial u}{\partial x} + \lambda e - \gamma \theta - \beta C \tag{7}$ 

$$\sigma_{yy} = 2\mu \frac{\partial v}{\partial y} + \lambda e - \gamma \theta - \beta C \tag{8}$$

$$\sigma_{xy} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \tag{9}$$

$$P = -\beta e + bC - a\theta \tag{10}$$

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{11}$$

Equations (1)–(3) thus reduce to

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} + (\lambda + \mu)\frac{\partial^2 v}{\partial x \partial y} + \mu \frac{\partial^2 u}{\partial y^2}$$
$$= \gamma \frac{\partial \theta}{\partial x} + \beta \frac{\partial C}{\partial x} + \rho \frac{\partial^2 u}{\partial t^2}$$
(12)

$$(\lambda + 2\mu)\frac{\partial^2 v}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 u}{\partial x \partial y} + \mu \frac{\partial^2 v}{\partial x^2}$$
$$= \gamma \frac{\partial \theta}{\partial y} + \beta \frac{\partial C}{\partial y} + \rho \frac{\partial^2 v}{\partial t^2}$$
(13)

$$\begin{pmatrix} k^* + k\frac{\partial}{\partial t} \end{pmatrix} \left( \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right)$$
  
=  $\frac{\partial^2}{\partial t^2} \left( \rho c_e \theta + a T_0 C + T_0 \gamma \frac{\partial u}{\partial x} + T_0 \gamma \frac{\partial v}{\partial y} \right)$ (14)

$$Db\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right) - Da\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right) - D\beta\left(\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial y^3} + \frac{\partial^3 v}{\partial y \partial x^2}\right) = \frac{\partial C}{\partial t} \quad (15)$$

Introducing the following non-dimensional variables

$$(x', y', u', v') = \frac{\eta}{c_1}(x, y, u, v), \quad t' = \eta t$$
  

$$\theta' = \frac{\gamma \theta}{\rho c_1^2}, \quad C' = \frac{\beta C}{\rho c_1^2}, \quad \eta = \frac{\rho c_e c_1^2}{k}$$
  

$$(\sigma'_{xx}, \sigma'_{xy}, \sigma'_{yy}) = \frac{1}{\rho c_1^2}(\sigma_{xx}, \sigma_{xy}, \sigma_{yy})$$
  

$$P' = \frac{P}{\beta}, \quad c_1^2 = \frac{\lambda + 2\mu}{\rho}$$
(16)

In terms of the non-dimensional quantities defined in Eq. (16), the above governing equations reduce to (dropping the dashed for convenience)

$$\sigma_{xx} = \frac{\partial u}{\partial x} + a_1 \frac{\partial v}{\partial y} - \theta - C \tag{17}$$

$$\sigma_{yy} = \frac{\partial v}{\partial y} + a_1 \frac{\partial u}{\partial x} - \theta - C \tag{18}$$

$$\sigma_{xy} = a_2 \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \tag{19}$$

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$$P = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + \xi_1 C - \xi_2 \theta \tag{20}$$

$$\frac{\partial^2 u}{\partial x^2} + (a_1 + a_2)\frac{\partial^2 v}{\partial x \partial y} + a_2\frac{\partial^2 u}{\partial y^2} = \frac{\partial \theta}{\partial x} + \frac{\partial C}{\partial x} + \frac{\partial^2 u}{\partial t^2} \quad (21)$$

$$\frac{\partial^2 v}{\partial y^2} + (a_1 + a_2) \frac{\partial^2 u}{\partial x \partial y} + a_2 \frac{\partial^2 v}{\partial x^2} = \frac{\partial \theta}{\partial y} + \frac{\partial C}{\partial y} + \frac{\partial^2 v}{\partial t^2} \quad (22)$$

$$\left(\varepsilon_{1}+\varepsilon_{2}\frac{\partial}{\partial t}\right)\left(\frac{\partial^{2}\theta}{\partial x^{2}}+\frac{\partial^{2}\theta}{\partial y^{2}}\right)$$
$$=\frac{\partial^{2}}{\partial t^{2}}\left(\theta+\varepsilon_{c}C+\varepsilon_{e}\frac{\partial u}{\partial x}+\varepsilon_{e}\frac{\partial v}{\partial y}\right) \qquad (23)$$

$$\xi_1 \left( \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} \right) - \xi_2 \left( \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) - \left( \frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 v}{\partial x^2 \partial y} + \frac{\partial^3 u}{\partial y^2 \partial x} + \frac{\partial^3 v}{\partial y^3} \right) = \xi_3 \frac{\partial C}{\partial t} \quad (24)$$

where,

$$a_{1} = \frac{\lambda}{\lambda + 2\mu}, \quad a_{2} = \frac{\mu}{\lambda + 2\mu}, \quad \varepsilon_{1} = \frac{k^{*}}{\rho c_{e} c_{1}^{2}}$$
$$\varepsilon_{2} = \frac{k\eta}{\rho c_{e} c_{1}^{2}}, \quad \varepsilon_{c} = \frac{aT_{0}\gamma}{\rho c_{e}\beta}, \quad \varepsilon_{e} = \frac{T_{0}\gamma^{2}}{\rho^{2} c_{e} c_{1}^{2}}$$
$$\xi_{1} = \frac{b\rho c_{1}^{2}}{\beta^{2}}, \quad \xi_{2} = \frac{a\rho c_{1}^{2}}{\gamma\beta}, \quad \xi_{3} = \frac{\rho c_{1}^{4}}{D\eta\beta^{2}}$$

The above equations are solved subjected to the initial conditions

$$u = v = \theta = C = 0, \quad t = 0,$$
  
 $\dot{u} = \dot{v} = \dot{\theta} = \dot{C} = 0, \quad t = 0$  (25)

The boundary conditions for the problem may be taken as

$$\theta(0, y, t) = \theta_o H(t) H(2l - |y|), \quad \sigma_{xx}(0, y, t) = 0$$
  
$$\sigma_{xy}(0, y, t) = 0, \quad P(0, y, t) = 0$$
(26)

where, H is the Heaviside unit step.

#### 4. FINITE ELEMENT FORMULATION

In this section, the governing equations of generalized thermoelastic diffusion (the Green and Naghdi theory) are summarized, followed by the corresponding finite element equations. In the finite element method, the displacement components u, v, temperature  $\theta$  and concentration C are related to the corresponding nodal values by

$$u = \sum_{i=1}^{m} N_{i} u_{i}(t), \quad v = \sum_{i=1}^{m} N_{i} v_{i}(t)$$
  

$$\theta = \sum_{i=1}^{m} N_{i} \theta_{i}(t), \quad C = \sum_{i=1}^{m} N_{i} C_{i}(t)$$
(27)

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where *m* denotes the number of nodes per element, and  $N_i$  are the shape functions. The eight-node isoparametric, quadrilateral element is used for displacement components, temperature and concentration calculations. The weighting functions and the shape functions coincide. Thus,

$$\delta u = \sum_{i=1}^{m} N_i \delta u_i, \quad \delta v = \sum_{i=1}^{m} N_i \delta v_i$$

$$\delta \theta = \sum_{i=1}^{m} N_i \delta \theta_i, \quad \delta C = \sum_{i=1}^{m} N_i \delta C_i$$
(28)

It should be noted that appropriate boundary conditions associated with the governing Eqs. (21)–(24) must be adopted in order to properly formulate a problem. Boundary conditions are either essential (or geometric) or natural (or traction) types. Essential conditions are prescribed displacements u, v, temperature  $\theta$  and concentration C while, the natural boundary conditions are prescribed tractions, heat flux and mass flux which are expressed as

$$\sigma_{xx}n_x + \sigma_{xy}n_y = \bar{\tau}_x, \quad \sigma_{xy}n_x + \sigma_{yy}n_y = \bar{\tau}_y$$

$$q_xn_x + q_yn_y = \bar{q}, \quad \eta_xn_x + \eta_yn_y = \bar{\eta}$$
(29)

where  $n_x$  and  $n_y$  are direction cosines of the outward unit normal vector at the boundary,  $\bar{\tau}_x$ ,  $\bar{\tau}_y$  are the given tractions values,  $\bar{q}$  is the given surface heat flux and  $\bar{\eta}$  is the given surface mass flux.

In the absence of body force, the governing equations are multiplied by weighting functions and then are integrated over the spatial domain  $\Omega$  with the boundary  $\Gamma$ . Applying integration by parts and making use of the divergence theorem reduce the order of the spatial derivatives and allows for the application of the boundary conditions. Thus, the finite element equations corresponding to Eqs. (21)–(24) can be obtained as

$$\left\{ \begin{array}{c} \frac{\partial \delta u}{\partial x} \sigma_{xx} + \frac{\partial \delta u}{\partial y} \sigma_{xy} \\ \frac{\partial \delta v}{\partial x} \sigma_{xy} + \frac{\partial \delta v}{\partial y} \sigma_{yy} \\ \left\{ \left( \varepsilon_{1} \frac{\partial \delta \theta}{\partial x} \frac{\partial \theta}{\partial x} + \varepsilon_{2} \frac{\partial \delta \theta}{\partial y} \frac{\partial^{2} \theta}{\partial t \partial y} \right) \\ \left( \frac{\partial \delta C}{\partial x} \frac{\partial P}{\partial x} + \frac{\partial \delta C}{\partial y} \frac{\partial P}{\partial y} \right) \end{array} \right\} d\Omega \\ + \int_{\Omega} \left\{ \begin{array}{c} \delta u \frac{\partial^{2} u}{\partial x^{2}} \\ \delta v \frac{\partial^{2} v}{\partial t^{2}} \\ \delta \theta \frac{\partial^{2}}{\partial t^{2}} \left( \theta + \varepsilon_{c} C + \varepsilon_{e} \frac{\partial u}{\partial x} + \varepsilon_{e} \frac{\partial v}{\partial y} \right) \\ & \varepsilon_{3} \frac{\partial C}{\partial t} \end{array} \right\} d\Omega$$

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**Fig. 1.** The temperature distribution  $\theta$  with distance x for different values of t at y = 0.5.

$$= \int_{\Gamma} \begin{cases} \delta u \bar{\tau}_{x} \\ \delta v \bar{\tau}_{y} \\ \delta \theta \bar{q} \\ \delta C \bar{\eta} \end{cases} d\Gamma$$
(30)

Substituting the constitutive relations (17)-(20) and Eqs. (28) and (29) into Eq. (30) leads

$M_1^e$	1	0	0	0 ]	ü <sup>e</sup>
0		$M^e_{22}$	0	0	$\ddot{v}^e$
$M_3^e$	1	$M^e_{32}$	$M^{e}_{33}$	<i>M</i> <sup><i>e</i></sup> <sub>34</sub>	$\ddot{\theta}^e$
0		0	0	0	$\ddot{C}^e$
0	0	0	0	] ( <i>ü<sup>e</sup></i>	)
0	0	0	0	$\dot{v}^e$	
0	0	$C_{33}^{e}$	0	$\left  \right  \dot{\theta}^{e}$	Ì
0	0	0	$C_{44}^{e}$	Ċe	J
	$M_{1}^{e}$ $M_{3}^{e}$ 0 0 0 0 0 0 0 0	$M_{11}^{e}$ 0 $M_{31}^{e}$ 0 0 0 0 0 0 0 0	$\begin{array}{cccc} M_{11}^{e} & 0 \\ 0 & M_{22}^{e} \\ M_{31}^{e} & M_{32}^{e} \\ 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 &$	$\begin{array}{ccccccc} M_{11}^e & 0 & 0 \\ 0 & M_{22}^e & 0 \\ M_{31}^e & M_{32}^e & M_{33}^e \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 &$	$\begin{bmatrix} M_{11}^{e} & 0 & 0 & 0 \\ 0 & M_{22}^{e} & 0 & 0 \\ M_{31}^{e} & M_{32}^{e} & M_{33}^{e} & M_{34}^{e} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 &$



Fig. 2. Horizontal displacement distribution u with distance x for different values of t at y = 0.5.



Fig. 3. Vertical displacement distribution v with distance x for different values of t at y = 0.5.

$$+ \begin{bmatrix} K_{11}^{e} & K_{12}^{e} & K_{13}^{e} & K_{14}^{e} \\ K_{21}^{e} & K_{22}^{e} & K_{23}^{e} & K_{24}^{e} \\ 0 & 0 & K_{33}^{e} & 0 \\ K_{41}^{e} & K_{42}^{e} & K_{43}^{e} & K_{44}^{e} \end{bmatrix} \begin{bmatrix} u^{e} \\ v^{e} \\ \theta^{e} \\ C^{e} \end{bmatrix} = \begin{cases} F_{1}^{e} \\ F_{2}^{e} \\ F_{3}^{e} \\ F_{4}^{e} \end{bmatrix} \end{pmatrix}$$
(31)

where me is the total number of elements. The coefficients in Eq. (31) are given below.

$$M_{11}^{e} = \int_{\Omega} [N]^{T} [N] d\Omega, \quad M_{22}^{e} = \int_{\Omega} [N]^{T} [N] d\Omega$$

$$M_{31}^{e} = \int_{\Omega} [N]^{T} [N] d\Omega, \quad M_{32}^{e} = \int_{\Omega} \varepsilon_{e} [N]^{T} [N] d\Omega$$

$$M_{33}^{e} = \int_{\Omega} \varepsilon_{e} [N]^{T} \left[\frac{\partial N}{\partial x}\right] d\Omega, \quad M_{34}^{e} = \int_{\Omega} \varepsilon_{e} [N]^{T} \left[\frac{\partial N}{\partial y}\right] d\Omega$$

$$C_{33}^{e} = -\int_{\Omega} \varepsilon_{2} [N]^{T} \left(\left[\frac{\partial^{2} N}{\partial x^{2}}\right] + \left[\frac{\partial^{2} N}{\partial y^{2}}\right]\right) d\Omega$$

$$C_{44}^{e} = \int_{\Omega} \xi_{3} [N]^{T} [N] d\Omega$$

$$K_{11}^{e} = \int_{\Omega} \left(\left[\frac{\partial N}{\partial x}\right]^{T} \left[\frac{\partial N}{\partial x}\right] + a_{2} \left[\frac{\partial N}{\partial y}\right]^{T} \left[\frac{\partial N}{\partial y}\right]\right) d\Omega$$

$$0.05 = \frac{1}{0.05} =$$

Fig. 4. Concentration distribution C with distance x for different values of *t* at y = 0.5.

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**Fig. 5.** Chemical potential distribution *P* with distance *x* for different values of *t* at y = 0.5.

$$K_{13}^{e} = \int_{\Omega} -\left[\frac{\partial N}{\partial x}\right]^{T} [N] d\Omega$$

$$K_{12}^{e} = \int_{\Omega} (a_{1} + a_{2}) \left[\frac{\partial N}{\partial x}\right]^{T} \left[\frac{\partial N}{\partial y}\right] d\Omega$$

$$K_{14}^{e} = \int_{\Omega} -\left[\frac{\partial N}{\partial x}\right]^{T} [N] d\Omega$$

$$K_{21}^{e} = \int_{\Omega} (a_{1} + a_{2}) \left[\frac{\partial N}{\partial y}\right]^{T} \left[\frac{\partial N}{\partial x}\right] d\Omega$$

$$K_{22}^{e} = \int_{\Omega} \left(\left[\frac{\partial N}{\partial y}\right]^{T} \left[\frac{\partial N}{\partial y}\right]^{T} [N] d\Omega$$

$$K_{22}^{e} = \int_{\Omega} \left(\left[\frac{\partial N}{\partial y}\right]^{T} \left[\frac{\partial N}{\partial y}\right] + a_{2} \left[\frac{\partial N}{\partial x}\right]^{T} \left[\frac{\partial N}{\partial x}\right]\right) d\Omega$$

$$K_{24}^{e} = \int_{\Omega} -\left[\frac{\partial N}{\partial y}\right]^{T} [N] d\Omega$$

$$K_{33}^{e} = \int_{\Omega} \left(\left[\frac{\partial N}{\partial x}\right]^{T} \left[\frac{\partial N}{\partial x}\right] + \varepsilon_{1} \left[\frac{\partial N}{\partial y}\right]^{T} \left[\frac{\partial N}{\partial y}\right]\right) d\Omega$$



**Fig. 6.** The distribution of stress component  $\sigma_{xx}$  with distance x for different values of t at y = 0.5.

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Fig. 7. The distribution of stress component  $\sigma_{xy}$  with distance x for different values of t at y = 0.5.



Symbolically, the discredited equations of Eq. (31) can be written as

$$M\ddot{d} + S\dot{d} + Kd = F^{\text{ext}} \tag{32}$$

where *M*, *S*, *K* and  $F^{\text{ext}}$  represent the mass, damping, stiffness matrices and external force vectors, respectively;  $d = [u \ v \ \theta \ C]^t$ . On the other hand, the time derivatives of



Fig. 8. The distribution of stress component  $\sigma_{yy}$  with distance x for different values of t at y = 0.5.

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Fig. 9. The temperature distribution  $\theta$  with distance x at t = 0.2 and y = 0.5.

the unknown variables have to be determined by Newmark time integration method (see Wriggers<sup>31</sup>).

## 5. NUMERICAL EXAMPLE

Following Sherief and Saleh<sup>13</sup> copper material is chosen for the purpose of numerical calculation.

$$\begin{split} \rho &= 8954 \text{ kg m}^{-3}, \quad \lambda = 7.76 \times 10^{10} \text{ kg m}^{-1} \text{ s}^{-2} \\ \mu &= 3.86 \times 10^{10} \text{ kg m}^{-1} \text{ s}^{-2}, \quad \varepsilon_2 = 0.25 \\ l &= 0.5, \quad c_e = 383.1 \text{ J kg}^{-1} \text{ k}^{-1}, \quad \alpha_t = 1.78 \times 10^{-5} \text{ k}^{-1} \\ \alpha_c &= 1.98 \times 10^{-4} \text{ m}^3 \text{ kg}^{-1}, \quad K = 386 \text{ w m}^{-1} \text{ k}^{-1} \\ T_o &= 293 \text{ k}, \quad D = 0.85 \times 10^{-8} \text{ kg s m}^{-3} \\ a &= 1.2 \times 10^4 \text{ m}^2 \text{ s}^{-2} \text{ k}^{-1}, \quad b = 0.9 \times 10^6 \text{ m}^5 \text{ kg}^{-1} \text{ s}^{-2} \end{split}$$

The graphically results for temperature distribution  $\theta$ , displacement components u, v, stress components  $\sigma_{xx}$ ,  $\sigma_{xy}$ ,  $\sigma_{yy}$ , concentration C and chemical potential P with distance x at y = 0.5 are shown in Figures 1–14. Figures 1–8 show that the effect of time (t = 0.05, 0.1, 0.15, 0.2) in the physical quantities under the thermoelastic diffusion



Fig. 10. Horizontal displacement distribution u with distance x at t = 0.2 and y = 0.5.



Fig. 11. Vertical displacement distribution v with distance x at t = 0.2 and y = 0.5.

(TED). Figure 1 shows the variation of temperature  $\theta$  with distance x and it indicates that temperature field has maximum value at the boundary and then decreases to zero. Figure 2 shows the variation of horizontal displacement u with respect to x at four different times and it indicates that when the surface of the half-space is taken to be traction free, and the thermal shock applied on the surface, the displacement at t = 0.2 shows a negative value at the boundary of the half space and it attains a stationary maximum value after some distance. Finally it decreases to zero value. The similar nature of variation is observed at t = 0.05, t = 0.1, t = 0.15. Figure 3 exhibits the variation of vertical displacement for different values of time in which we observed that, significant difference in the value of displacement is noticed for the different value of t. Figure 4 show the variation of concentration C with distance x for four different values of t. It is clear that the time has an increasing effect. Figures 5-8 shows the variations of chemical potential P and stress field with respect to x for different values of t, in which we observed that, the chemical potential P, stresses  $\sigma_{xx}$  and  $\sigma_{xy}$  are zero at x = 0 which satisfies the boundary conditions of the

 $\begin{array}{c}
0.4 \\
0.3 \\
0.2 \\
0.1 \\
-0.2 \\
-0.3 \\
-0.4 \\
-0.5 \\
0 \\
0.5 \\
1 \\
1.5 \\
2
\end{array}$ TED

Fig. 12. The distribution of stress component  $\sigma_{xx}$  with distance x at t = 0.2 and y = 0.5.

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Fig. 14. The distribution of stress component  $\sigma_{yy}$  with distance x at t = 0.2 and y = 0.5.

problem. Figures 9–14 show the variation of the physical quantities with space x at t = 0.2 under two types, thermoelastic with diffusion (TED) and thermoelastic without diffusion (TE). It easily to see that, the diffusion has a significant effect on the physical quantities.

## NOMENCLATURE

- $\rho$  Mass density
- T Absolute temperature
- C Mass concentration
- $\lambda, \mu$  Lame's constants
  - $u_i$  Components of displacement vector
  - $\sigma_{ii}$  Stresses components
  - $c_e$  Specific heat at constant strain
  - $\theta$  Temperature increment
  - $T_0$  Reference temperature chosen so that  $|(T T_0)/T_0| \ll 1$
  - $k^*$  Material characteristic of the theory

- k Thermal conductivity
- P Chemical potential per unit mass
- $\delta_{ij}$  Kronecker delta
- *d* Thermodiffusion constant
- a Measure of thermodiffusion effect
- b Measure of diffusive effect
- $\gamma = (3\lambda + 2\mu)\alpha_t$
- $\beta = (3\lambda + 2\mu)\alpha_c$
- $\alpha_t$  Coefficient of linear thermal expansion
- $\alpha_c$  Coefficient of linear diffusion expansion.

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